Simplification of Nonlinear Indicial Response Models: Assessment for the Two-Dimensional Airfoil Case

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Simplifications to the functional form of the nonlinear indicial response model and application to the aerodynamic response due to arbitrary motion inputs are discussed. Numerical results for a thin two-dimensional airfoil with wake distortion nonlinearities are used to control and justify the assumptions employed. Useful relationships between indicial response parameters and steady-state oscillatory force and moment data are obtained by representing the indicial response with a Taylor series (in terms of onset motion parameters) and approximating the superposition integral with an asymptotic expansion. In particular, the appearance of certain harmonics and their variation with frequency, amplitude, and mean angle of attack are traced to specific onset parameters. A nonlinear parameter identification procedure is proposed by which active onset parameters may be determined from experimental data. Restrictions imposed by using the asymptotic expansion are also examined. Nonlinear stability derivatives are shown to be related to specific indicial response characteristics.

Nomenclature

A	= amplitude of harmonic oscillation in α, rad
C_L	= section lift coefficient, lift/qc
$C_{L_{am}}$	= apparent mass derivative, 1/rad
C_{L_0}	= mean section lift coefficient for oscillatory motion
C_{L_α}	= nonlinear indicial response, section lift due to α
c	= chord length, ft
F	= deficiency function, difference between steady-state and transient indicial responses, 1/rad
F_0	= linear deficiency function evaluated at $\alpha = 0$, 1/rad
F_2	= nonlinear deficiency function, $\partial^2 F/\partial\alpha^2$ evaluated at $\alpha = 0$, 1/rad^3
g_0	= superposition integral linear onset parameter, $= \dot{\alpha}$
g_2	= superposition integral nonlinear onset parameter, $= \alpha^2\dot{\alpha}$
k	= reduced frequency, $= \omega c/2U_\infty$
q	= dynamic pressure, psf
R_n	= remainder after n terms of partial sum
t	= time, nondimensionalized by $2U_\infty/c$
U_∞	= freestream velocity, ft/s
α	= angle of attack, rad
$\dot{\alpha},\ddot{\alpha}$	= first and second derivatives of α with respect to t, rad
α_0	= mean angle of attack for harmonic motion, rad
τ	= nondimensional time at step onset
τ_1	= elapsed nondimensional time from onset, $= t - \tau$
ω	= circular frequency, rad/s

Introduction

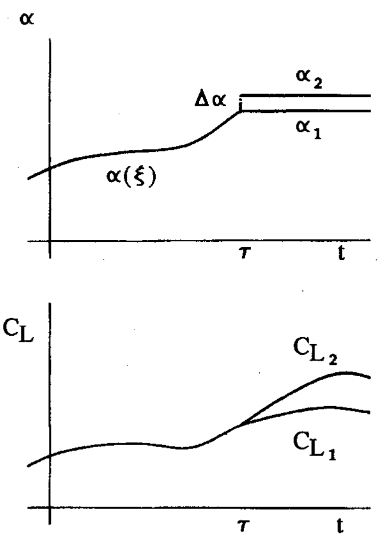
LARGE-AMPLITUDE oscillatory force and moment data typically exhibit nonlinear dependencies on amplitude and frequency. Often these effects can be attributed to hysteresis; sometimes "simple" nonlinear variations in either (or both) static or dynamic force and moment data are the cause. In either case, aerodynamic response modeling for flight mechanics analyses is a difficult problem.

The current work focuses on establishing relationships between steady-state oscillatory force and moment data and models capable of accounting for some of these nonlinear effects. Limits of applicability for these models is also explored. The basis for this study is nonlinear aerodynamic modeling work reported by Tobak and Chapman[1] and Tobak and Schiff.[2]

The modeling concepts apply equally well to any of the six force and moment components and to any motion input, including control deflection; however, lift response due to angle of attack is used throughout to illustrate the concepts.

Nonlinear Indicial Responses

The nonlinear indicial response is the most generally applicable modeling concept. It is defined in terms of two motions as shown in the following sketch:



where $\alpha(\xi)$ is the "reference motion," defined for $-\infty < t \le \tau$; α_1 consists of $\alpha(\xi)$, for $t \le \tau$, and is held constant at $\alpha(\tau)$ for $t > \tau$; and α_2 consists of $\alpha(\xi)$, for $t \le \tau$, but jumps instantaneously to $\alpha(\tau) + \Delta\alpha$ for $t > \tau$. The nonlinear indicial lift response is the limit, as step height $\Delta\alpha$ approaches zero, of the difference between the corresponding lift time histories.

Evidence supporting the need for such a definition is given in Figs. 1–3, taken from Graham's[3] tow-tank experiments

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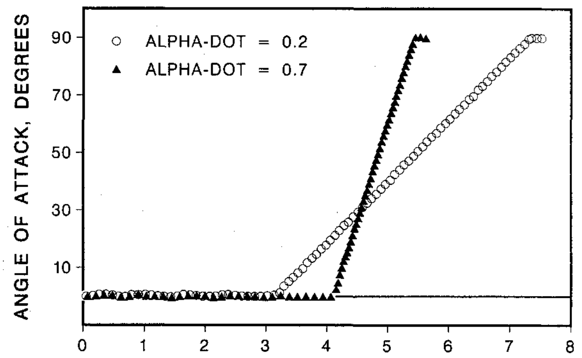

Fig. 1 Reference motions.

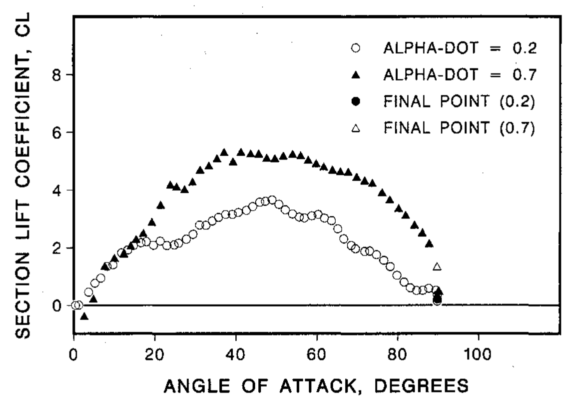

Fig. 2 Lift time history.

using a NACA 0015 airfoil. The point is that motion history effects can have a profound influence on the resulting flowfields, even when there are identical onset conditions. In fact, history effects are completely reflected in the wake structure.

Consider two angle-of-attack time histories, as shown in Fig. 1, which are interpreted as two distinct reference motions. (Graham[3] investigated "poststall maneuvers" that returned to zero angle of attack after holding at high α; however, the point is better illustrated by examining only a portion of the complete motion.) Flowfields at the termination of each maneuver ($\alpha = 90$ deg) are shown in Figs. 3. The flow is left to right and the view is spanwise from the root, slightly above and behind the wing section. Thus, only the upper surface of the wing section is visible; the leading edge is to the top of the picture. For the rapid pitch-up (alpha dot = 0.7), the "dynamic stall" vortex is less diffuse, more tightly wrapped, and closer to the airfoil upper surface than for alpha dot = 0.2. Note also that at alpha dot = 0.7 the starting vortex associated with the onset of the pitch-up motion (lower right) has not had time to be swept downstream and can still exert an appreciable influence on airfoil response. Furthermore, if α is held constant at this point, these distinctly different vortex systems will also exhibit unique diffusion and convection properties. Correspondingly large differences in lift are observed (Fig. 2). Clearly, motion history effects can be important. Therefore, the reference motion is required to establish appropriate flow conditions from which to measure the corresponding indicial response. The α_2 motion then establishes *the change in response due to a small perturbation in* α.

As shown by Tobak and Chapman,[1] the dependence of nonlinear indicial responses on reference motion requires that they be expressed mathematically as a functional, i.e.,

$$C_{L_\alpha}[\alpha(\xi);t,\tau] = \lim_{\Delta\alpha \to 0} \frac{C_L[\alpha_2(t)] - C_L[\alpha_1(t)]}{\Delta\alpha} \tag{1}$$

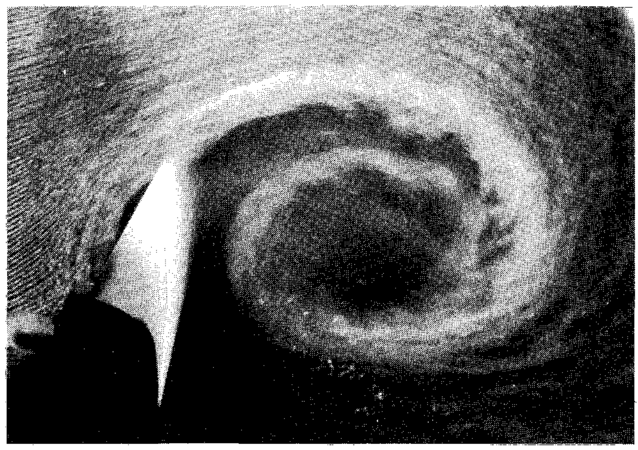

Fig. 3a Flow visualization, alpha dot = 0.2.

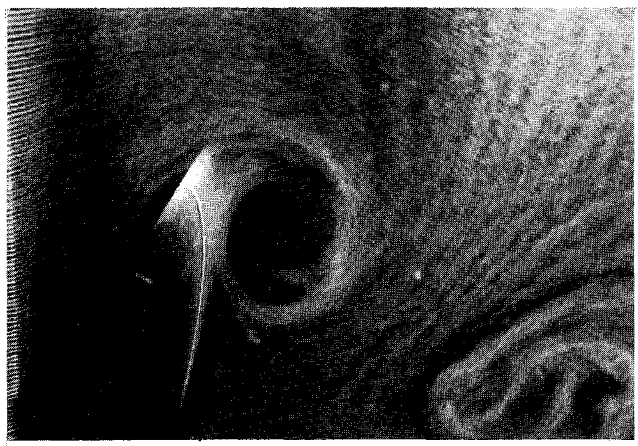

Fig. 3b Flow visualization, alpha dot = 0.7.

where the step onset is at $t = \tau$. The functional dependency on prior motion $\alpha(\xi)$ distinguishes the nonlinear indicial response from its linear counterpart.

Equation (1) defines the Fréchet derivative of the functional $C_L[\alpha_1(t)]$, as noted by Tobak and Chapman.[1] They also suggest that bifurcations of physically realizable (asymptotically stable to small perturbations) steady-state solutions corresponding to α_1 are signaled by loss of Fréchet differentiability. Such occurrences are of considerable interest to the study of hysteresis effects.

Possible simplifications to the functional representation for indicial responses have been suggested by Tobak and Schiff.[2] Suppose that the motion $\alpha(\xi)$ is analytic (in the strict mathematical sense) over the interval $-\infty < t < \tau$. In this case, $\alpha(\xi)$ may be replaced by its Taylor series expansion about $\xi = \tau$. Therefore,

$$C_{L_\alpha}[\alpha(\xi);t,\tau] = C_{L_\alpha}[t,\tau;\alpha(\tau),\dot{\alpha}(\tau),\ddot{\alpha}(\tau),...] \tag{2}$$

where the independent variables $\alpha(\tau),\dot{\alpha}(\tau),\ddot{\alpha}(\tau),...$ are the coefficients of the Taylor expansion. On physical grounds, the distant past is expected to be less important to the step response than motion characteristics just prior to onset, suggesting, perhaps, that only a few Taylor series coefficients need to be retained.

Simplifications based on unsteady aerodynamic characteristics for thin two-dimensional airfoils in an inviscid and incompressible fluid are studied in the following section. This approach allows the nonlinear indicial response model to be examined in a context where assumptions may be controlled and justified. However, results reported herein are not applicable to the extreme conditions evident in the example just presented. Notably large-scale separated flows with corresponding time-dependent equilibrium states are excluded.

Nonlinear thin-airfoil characteristics were obtained by using NLWAKE, a computer code developed by Scott and McCune.[4] NLWAKE provides numerical solutions for a nonlinear version of Wagner's integral equation, which relates the quasisteady bound vorticity to wake vorticity. To do this, the shed vorticity is discretized and the wake allowed to distort under the influence of both bound vorticity and other wake elements. Since NLWAKE disallows separation, only nonlinear wake distortion effects are included. NLWAKE cannot be used to investigate hysteresis effects.

Two-Dimensional Airfoil with Wake Distortion

Nonlinear step responses computed with NLWAKE are shown in Figs. 4 and 5. The required Fréchet derivatives were computed numerically based on positive and negative step heights of 0.1 deg. Figure 4 shows indicial responses for constant angle of attack prior to onset. In Fig. 5, $\alpha(\xi)$ is ramped at constant rate (positive and negative) to the same onset α values. Motion history effects caused by wake distortion have an appreciable influence on the indicial response. However, history effects predicted by NLWAKE correlate very well with onset angle of attack; onset rate having a negligible influence up to quite large values. Thus, for the two-dimensional airfoil in the absence of separation, indicial response functionals may be adequately approximated by a function only of $\alpha(\tau)$ and elapsed time from onset $t-\tau$. This approximation is used in the following development although separation effects are probably important at lower onset rates than shown in Fig. 5.

NLWAKE cannot predict nonlinearities in the static lift-curve slope. However, for completeness, this possibility is included, although coupling between quasisteady characteristics and indicial response time history cannot be explored. Also, there are situations (e.g., following a Hopf bifurcation) where the response is time dependent even in the limit as elapsed time (since step initiation) approaches infinity. However, a steady-state condition is assumed to exist. In these cases, it is convenient to introduce the "deficiency function" F:

$$F[\alpha(\tau),t-\tau] = C_{L_\alpha}[\alpha(\tau),\infty] - C_{L_\alpha}[\alpha(\tau),t-\tau]$$

where the first term on the right side is the steady-state lift-curve slope,

$$C_{L_\alpha}[\alpha(\tau),\infty] = \lim_{t-\tau\to\infty} C_{L_\alpha}[\alpha(\tau),t-\tau]$$

and the indicial response is expressed as a function of onset angle of attack and elapsed time as suggested above. Thus Eq. (2) becomes

$$\begin{aligned} C_{L_\alpha}[\alpha(\xi);t-\tau] &\simeq C_{L_\alpha}[\alpha(\tau),t-\tau] \\ &= C_{L_\alpha}[\alpha(\tau),\infty] - F[\alpha(\tau),t-\tau] \end{aligned} \tag{3}$$

In addition, for uncambered airfoils (consistent with NLWAKE restrictions), both the deficiency function and static lift-curve slope are even functions of α. Furthermore, except for possible bifurcation points, both terms on the right side of Eq. (3) are expected to be analytic. Thus, expanding them in Taylor series about zero of angle of attack, retaining only even powers of alpha, Eq. (3) becomes

$$\begin{aligned} C_{L_\alpha}[\alpha(\xi);t-\tau] &\simeq C_{L_\alpha}[\alpha(\tau),t-\tau] \\ &= C_{L_\alpha}(0,\infty) + 0.5\frac{\partial^2}{\partial\alpha^2}[C_{L_\alpha}(0,\infty)]\alpha^2(\tau) + \cdots \\ &\quad - F_0(0,t-\tau) - 0.5F_2(0,t-\tau)\alpha^2(\tau) + \cdots \end{aligned} \tag{4}$$

where

$$F_0(0,t-\tau) = F[\alpha(\tau),t-\tau]\big|_{\alpha(\tau)=0}$$

and

$$F_2(0,t-\tau) = \frac{\partial^2}{\partial\alpha^2}\{F[\alpha(\tau),t-\tau]\}_{\alpha(\tau)=0}$$

$C_{L_\alpha}(0,\infty)$ and $F_0(0,t-\tau)$ are the classical linear terms because they are independent of onset conditions. Note that if additional onset parameters are included, multivariate Taylor series will be required and mixed partial derivatives will appear.

F_2 was computed from NLWAKE step responses at steady-state onset conditions of ± 1.0 deg and is shown in Fig. 6. Figure 4 shows the comparison between steps initiated at 4.0-deg alpha as calculated directly from NLWAKE and as

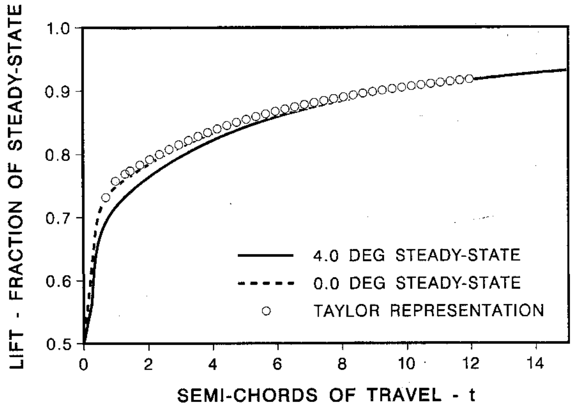

Fig. 4 Onset angle-of-attack effect on step response.

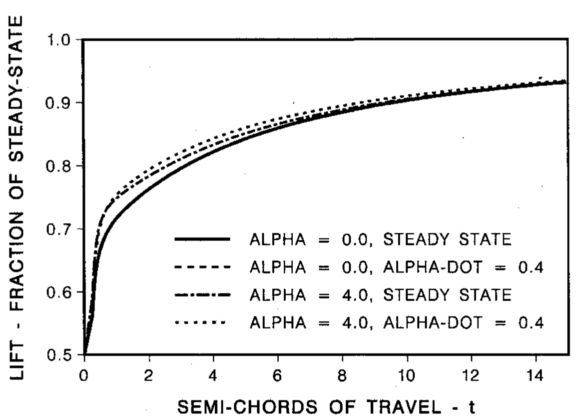

Fig. 5 Onset rate effect on indicial response.

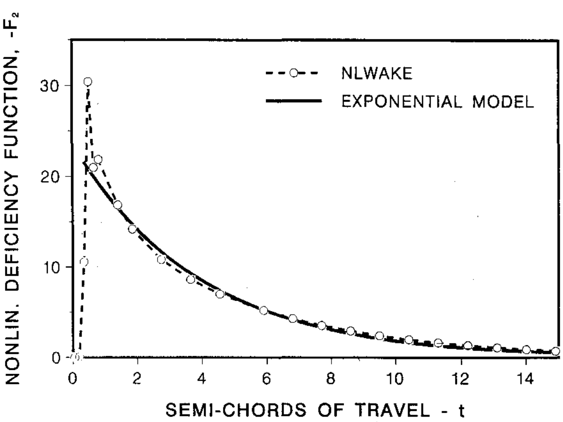

Fig. 6 Alpha-squared contribution to step response.

given by Eq. (4), dropping all terms higher than second order and retaining only the linear lift-curve slope. This is nearly an exact representation, and higher-order terms are not needed.

When Eq. (4) is used to construct lift responses to arbitrary inputs, as discussed in the following section, various integrals of F_0 and F_2 will be required. To this end, it is useful to have analytical models for each. Exponential approximations for the two-dimensional airfoil linear deficiency function abound in the literature. Jones' expression, as given by Fung,[5] is used herein. An exponential approximation for F_2, based on a least-squares fit to NLWAKE data, is shown in Fig. 6. These approximations are of the form:

$$F_0 \simeq a_1 e^{-a_2\tau_1} + (\pi - a_1)e^{-a_3\tau_1} \tag{5a}$$

$$F_2 \simeq b_1(1 - e^{-b_2\tau_1})e^{-b_3\tau_1} \tag{5b}$$

where $a_1 = 1.037$ (1/rad), $a_2 = 0.0455$, $a_3 = 0.300$, $b_1 = -148.3$ (1/rad^3), $b_2 = 24.9$, and $b_3 = 0.254$.

Equation (4) is the nonlinear indicial response form used throughout the remainder of the paper; principal limitations on its use are the following:

1) Reference motions must be analytic. Indicial response functionals can then be represented as functions of onset motion parameters.

2) The indicial responses are assumed to approach a steady-state condition as elapsed time since onset becomes large.

3) In quantitative results, terms above second order in α have been neglected based on comparisons with NLWAKE output (see Fig. 4). This has been justified only for attached-flow conditions and results are implicitly restricted to a relatively small angle-of-attack range.

4) The deficiency function and the static lift-curve slope have been expanded in Taylor series about alpha = 0 and a symmetric airfoil is considered; only even terms are retained. This leads to a simpler presentation and clearly shows that the classical linear theory is contained as a special case. Both terms can be expanded about nonzero values of alpha, if desired.

Response to Arbitrary Motion

Superposition Integral

As shown by Tobak and Chapman,[1] responses to arbitrary motion inputs can be calculated by using a generalized superposition integral. If a bifurcation in the steady-state response occurs at $t = \tau_c$, the integral has the form

$$C_L(t) = C_L[t,\alpha(0)] + \lim_{\varepsilon \to 0}\left\{\int_0^{\tau_c - \varepsilon} C_{L_\alpha}[\alpha(\xi);t,\tau]\frac{d\alpha}{d\tau}\,d\tau + \int_{\tau_c+\varepsilon}^{t} C_{L_\alpha}[\alpha(\xi);t,\tau]\frac{d\alpha}{d\tau}\,d\tau + \Delta C_L(t,\alpha_c)\right\} \tag{6}$$

where

$$\Delta C_L(t;\alpha_c) = C_L[\alpha(\xi);t,\tau_c + \varepsilon] - C_L[\alpha(\xi);t,\tau_c - \varepsilon]$$

Thus, the superposition integral is split to allow the solution to change discretely to a new equilibrium state (and to avoid the singularity). Since NLWAKE cannot support an investigation of hysteresis effects, it is assumed that no bifurcations take place over the time interval 0 to t. In this case, integration can proceed directly from $\tau = 0$ to $\tau = t$ and $\Delta C_L = 0$.

Simplified Forms

Approximating the nonlinear functional by Eq. (4) (dropping terms above second order) and changing the variable of integration to elapsed time from onset $\tau_1 = t - \tau$, Eq. (6) becomes

$$C_L(t) = C_L[t,\alpha(0)] + C_{L_\alpha}(0,\infty)[\alpha(t) - \alpha(0)] + C_{L_{\alpha\alpha\alpha}}\frac{\alpha^3(t) - \alpha^3(0)}{6} - \int_0^t F_0(0,\tau_1)g_0(t - \tau_1)\,d\tau_1 - 0.5\int_0^t F_2(0,\tau_1)g_2(t-\tau_1)\,d\tau_1 \tag{7}$$

where

$$C_{L_{\alpha\alpha\alpha}} = \frac{\partial^2}{\partial\alpha^2}[C_{L_\alpha}(0,\infty)], \quad g_0 = \dot{\alpha}, \quad g_2 = \alpha^2\dot{\alpha}$$

If Eq. (7) is used for dynamic analyses, the equations of motion become integro-differential equations because g_0 and g_2 appear under the integrals. Further simplification would be needed to avoid this complication. For linear systems, Etkin[6] proposed using "aerodynamic transfer functions" in conjunction with the Laplace transform of the equations of motion. One possible approach to the nonlinear problem is to represent Eq. (7) in the frequency domain by using higher-order Laplace transforms and George's association of variables technique (see Ref. 7). There would be, then, a potentially manageable algebraic problem with advantages similar to Etkin's linear system approach.

The objective here is limited to developing relationships between indicial response parameters and the response to arbitrary, but prescribed, motion. To this end, the integrals

$$\int_0^t F_i(\tau_1)g_i(t-\tau_1)\,d\tau_1; \qquad i = 0,2$$

may be integrated recursively by parts to give

$$\int_0^t F_i(\tau_1)g_i(t-\tau_1)\,d\tau_1 = g_i(0)I_{i1}(t) - g_i(t)I_{i1}(0) + \dot{g}_i(0)I_{i2}(t) - \dot{g}_i(t)I_{i2}(0) + \cdots - \frac{d^{n-1}}{dt^{n-1}}[g_i(t)]I_{in}(0) + R_{i,n} \tag{8a}$$

where

$$I_{in}(t) = \underbrace{\int\cdots\int}_{n\ \text{times}} F_i(\tau_1)\,d\tau_1\cdots d\tau_1\bigg|_{\tau_1 = t} \tag{8b}$$

$$I_{in}(0) = \underbrace{\int\cdots\int}_{n\ \text{times}} F_i(\tau_1)\,d\tau_1\cdots d\tau_1\bigg|_{\tau_1 = 0} \tag{8c}$$

$$R_{i,n} = \int_0^t I_{in}(\tau_1)\frac{d}{dt^n}[g_i(t-\tau_1)]\,d\tau_1 \tag{8d}$$

Differentiability of g_0 and g_2 imposes no new restrictions since Eq. (2) requires $\alpha(t)$ to have derivatives of all orders.

Finally, a series representation for the lift response to an arbitrary, but *specified*, motion input (in the absence of bifurcation) is given by substituting Eq. (8a) back into Eq. (7):

$$C_L = C_{L_{\text{am}}}\dot{\alpha}(t) + C_{L_\alpha}(0,\infty)\alpha(t) + C_{L_{\alpha\alpha\alpha}}\frac{\alpha^3(t)}{6} + \sum_{n=0}^{\infty}\left(\mathrm{I}_{n+1}(0)\left[\frac{d^n g_0}{dt^n}(t)\right] - \mathrm{I}_{n+1}(t)\left[\frac{d^n g_0}{dt^n}(0)\right] + 0.5\left\{\mathrm{J}_{n+1}(0)\left[\frac{d^n g_2}{dt^n}(t)\right] - \mathrm{J}_{n+1}(t)\left[\frac{d^n g_2}{dt^n}(0)\right]\right\}\right) \tag{9}$$

where a steady-state initial condition has been assumed and

$$\mathrm{I}_n = I_{0n} \qquad \text{and} \qquad \mathrm{J}_n = I_{2n}$$

Note that an apparent mass term $C_{L_{am}}\dot{\alpha}(t)$ must be included to account for the noncirculatory part of the lift response. The incompressible apparent mass reaction is instantaneous; therefore, the indicial response requires an impulse at step onset. Neither F_0 nor F_2 includes the impulse. However, it can be included as a Dirac delta function embedded in the indicial response representation.[8] Subsequent integration across the delta function produces an apparent mass term proportional to the instantaneous value of alpha dot. In the linear compressible case, Leishman[9] gives exponential models for the noncirculatory terms.

Note also on the right side of Eq. (9) the top line (apparent mass excluded) gives the quasisteady response. The last two lines account for the transition from the initial quasisteady state to the steady-state condition; i.e., $I_n(t)$ and $J_n(t)$ describe the transient response and vanish for large t. $I_n(0)$ and $J_n(0)$ provide the dynamic response.

If a bifurcation in possible steady-state solutions occurs inside the time interval, the superposition integral must be split in accordance with Eq (6). The integration procedure outlined earlier remains valid; the impact will be felt in two ways. First, by definition, static lift-curve slope terms will differ on each side of the bifurcation (corresponding changes in the deficiency functions are expected). Second, because τ_c appears in the limits of integration (and in ΔC_L), terms like $g_i(t-\tau_c)I_{i1}(\tau_c)$ will appear. Since, for arbitrary motion, τ_c is not known a priori, the time-domain solution becomes awkward for flight mechanists. Aerodynamic reaction models, as proposed by Hanff,[10] avoid this difficulty.

Properties of the Series Representation

Equation (9) has the distinct advantage of separating motion variables from indicial function characteristics. More specifically, the integrals are now independent of motion input; i.e., only multiple integrals of F_0 and F_2 are required. This property leads directly to the desired relationships between specific indicial response onset parameters, stability derivatives, and steady-state response to oscillatory motion inputs. However, simplification has been obtained at the cost of generality. Properties of both series (linear effects given in terms of I_n and nonlinear effects given in terms of J_n) are examined later.

First, a special word of caution is in order. It may be tempting to use Eq. (9), retaining higher-order terms to improve accuracy, directly in dynamic analyses. Even in the linear case, this can lead to disastrously false results. The reason is that expansion of the integrals alters the nature of the characteristic equation, changing it from transcendental to polynomial. Each additional term contains higher-order time derivatives and therefore increases the polynomial's degree. Inevitably, extraneous roots are introduced. Spurious roots can occur in awkward places (well into the right-half plane, for example) even if the approximation is quite good within the radius of convergence. For a discussion of this problem see Ref. 11 and the references cited therein. To reiterate, Eq. (9) is introduced simply as an analytical tool for studying relationships among indicial response characteristics and responses to *prescribed* motions such as those encountered in wind-tunnel testing.

Both series in Eq. (9) are asymptotic expansions [of the left side of Eq. (8a)] with respect to sequences defined by taking successive time derivatives of g_0 and g_2. This implies that they are valid for "sufficiently slow" motions, as will be shown for the special case of harmonic oscillations about a constant mean angle of attack. The motion also has an arbitrary phase angle relative to a reference signal:

$$\alpha(t)=\alpha_0+A\cos(kt+\phi) \tag{10a}$$

$$g_0(t)=-Ak\sin(kt+\phi) \tag{10b}$$

$$g_2(t)=-k\{B_3\sin[3(kt+\phi)]+B_2\sin[2(kt+\phi)] + B_1\sin(kt+\phi)\} \tag{10c}$$

where

$$B_1=[A(4\alpha_0^2+A^2)]/4,\quad B_2=A^2\alpha_0,\quad B_3=A^3/4$$

Now consider the sequences of functions

$$g_{i,n}=\frac{d^n}{dt^n}[g_i(t)];\qquad i=0,2 \tag{11}$$

where $n=0,1,2,3,\ldots$.

The right side of Eq. (8a) is an asymptotic expansion if $g_{i,n}$ is an asymptotic sequence and $R_{i,n}$ is $o(g_{i,n-1})$; that is, if

$$\lim_{k\to 0}\frac{g_{i,n+1}}{g_{i,n}}=0\quad\text{and}\quad\lim_{k\to 0}\frac{R_{i,n}}{g_{i,n-1}}=0\qquad\text{for all } n \tag{12}$$

General expressions for $g_{0,n}$ and $g_{2,n}$ can be obtained by putting Eqs. (10b) and (10c) into Eq. (11). Similarly, I_n and J_n are evaluated by repeated integration of Eqs. (5a) and (5b) in accordance with Eqs. (8b) and (8c). Proof that the series satisfy Eq. (12) is then straightforward. For example, when n is odd, $g_{2,n-1}$ is

$$k^n(3^{n-1}B_3\sin 3\Omega+2^{n-1}B_2\sin 2\Omega+B_1\sin\Omega)$$

and the dominant terms of $R_{2,n}$ vary with frequency as

$$\frac{k^{n+1}\cos 3\Omega}{k^2+f_1},\quad\frac{k^{n+1}\cos 2\Omega}{k^2+f_2},\quad\frac{k^{n+1}\cos\Omega}{k^2+f_3}$$

where

$$\Omega=kt+\phi$$

Note that f_1, f_2, and f_3 are independent of k. Since Ω approaches ϕ in the limit, the nonlinear terms are easily shown to satisfy Eq. (12). A similar argument holds for even values of n.

The most important property of asymptotic expansions is that their partial sums have an error of the same order as the first term omitted. For all values of n, Eq. (12) guarantees that the error can be made arbitrarily small as k approaches zero. However, the quality of the approximation depends on the convergence properties of the series for fixed k. If partial sums decrease initially, useful approximations can be obtained even if the series ultimately diverge. Thus, the question of establishing a frequency limit for practical applications is best answered by examining the behavior of the first few terms.

To this end, steady-state lift response characteristics for an airfoil oscillating about zero mean angle of attack were computed. Motion variables are given by Eqs. (10a–c) with $\alpha_0=0$. There can be no response at the first harmonic frequency $2k$ since $B_2=0$ in this case.

Approximate lift responses were computed from Eq. (9) with $I_n(0)$ and $J_n(0)$ defined by Eqs. (5a), (5b), and (8c). Both $I_n(t)$ and $J_n(t)$ were set to zero since only the steady-state solution was desired. Corresponding exact steady-state solutions were computed directly from Eq. (7) also using Eqs. (5a) and (5b) for F_0 and F_2, respectively. Nonlinearity in the quasisteady lift curve was neglected in both cases.

Partial sums of the series expansions, normalized by corresponding exact solutions, are presented in Figs. 7–9. In keeping with traditional dynamic testing practice, in-phase (cosine) and out-of-phase (sine) components are presented separately because they can be associated with static and dynamic stability derivatives. (Splitting the series is permissible, as the sum of two asymptotic expansions is also asymptotic, provided they are defined with respect to the same sequence.)

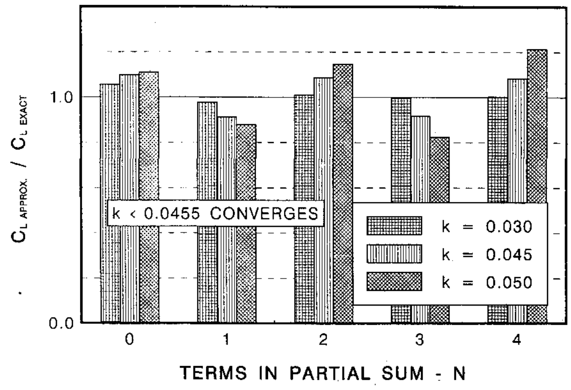

Fig. 7 Linear in-phase series convergence.

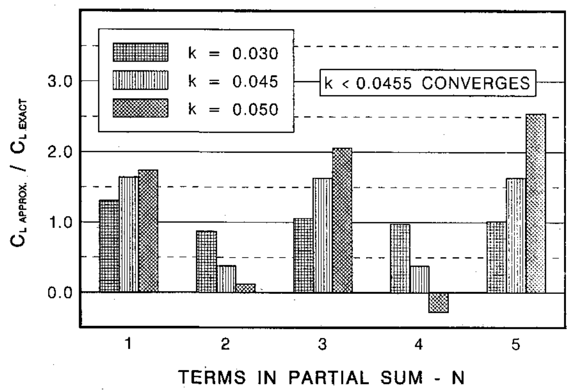

Fig. 8 Linear out-of-phase series convergence.

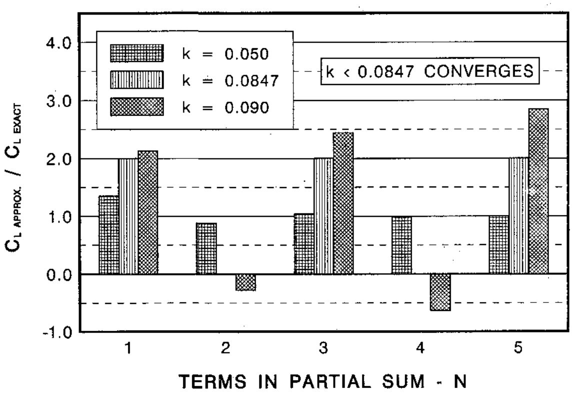

Fig. 9 Second harmonic in-phase series convergence.

Figure 7 shows partial sums representing the linear in-phase component as a function of frequency and the number of terms retained. Only the static lift-curve slope is included at $N = 0$, while $I_2 g_{0,1}$ is added for $N = 1$, and so on. As determined by the ratio test, series for the linear components converge for $k < a_2 = 0.0455$. Thus, characteristics just below, at, and just above the convergence frequency are illustrated. Asymptotic behavior is evident; accuracy is improved at any N as frequency decreases. At frequencies above the convergence frequency, however, the partial sums diverge immediately. If only the quasisteady term is retained, common practice for rigid-body dynamics, errors on the order of 10 and 11% will be incurred at reduced frequencies of 0.045 and 0.050, respectively. Finally, the static lift-curve slope is within 6% and convergence is rapid (to about 2% with one additional term) at the lowest frequency of 0.03.

Convergence properties for the linear out-of-phase components are shown in Fig. 8 and are quite similar to the linear in-phase component. Greater *percentage* errors for the initial term (31, 64, and 74% in order of increasing frequency) are expected since there is no quasisteady term common to both exact and approximate results.

Properties for the nonlinear in-phase contribution at $3k$ are displayed in Fig. 9. In this case, divergence occurs at frequencies above $3k = b_3 = 0.254$. Therefore, computations were carried out at reduced frequencies of 0.050, 0.0847, and 0.090 to include the highest frequency shown for the linear case and to overlap the convergence frequency. Again, divergence is immediate outside the radius of convergence. At $k = 0.050$ convergence is relatively fast, to within about 4% in three terms; however, the initial error is surprisingly high (35%) considering that this frequency is well removed from the divergence boundary of 0.0847.

Similar results could be shown for the remaining nonlinear contributions; but the point is that the series diverge for frequencies greater than *the slowest varying exponential involved in the corresponding indicial response*. That is, if for large t, the indicial response varies as e^{-at}, then the convergence frequency is given in terms of the indicial response time constant by

$$kT = 1$$

where

$$T = 1/a$$

For responses at harmonics of the forcing function, the harmonic frequency is to be used.

Relationship to Stability Derivatives

As Etkin[12] has shown, the conventional stability derivative representation can be derived easily from the premise that the aerodynamic reactions are functionals of the vehicle state variables. Two assumptions are required:

1) The motion is an analytic function of time.

2) Aerodynamic reactions are *analytic* functions of the instantaneous state variables and their time derivatives.

The first assumption allows the functional to be replaced by an ordinary function of the state variables and their derivatives; the second permits a multivariate Taylor series expansion of *the function* in terms of its arguments. The coefficients of the series are the classic stability derivatives. Only the linear terms are usually retained for small perturbation analyses; however, nonlinear derivatives are sometimes included. Alternatively, nonlinear effects can be included by treating the derivatives as functions of the variables.

Similarities between Etkin's development and the derivation of Eq. (4) are apparent. The significant difference is that the latter is an approximation to a functional representing the nonlinear indicial response rather than one representing the *total* response. In the absence of bifurcation, putting Eq. (4) into the superposition integral gives the total response to an analytic motion input, Eq. (7). In this section, the asymptotic approximation to Eq. (7), given by Eq. (9), is used to briefly examine the implicit limits of the stability derivative approach. First, to gain physical insight, Eq. (9) is used to replicate a previous result derived by Tobak and Schiff[2] for the case of slowly varying harmonic oscillations about constant mean values.

Recall that the infinite series in Eq. (9) are expansions for the left side of Eq. (8a) (with $i = 0$ and $i = 2$, respectively) and that $I_n(t)$ and $J_n(t)$ tend to zero as t goes to infinity. Thus, taking a sufficiently large t, say t_a, and retaining only the leading terms ($n = 0$) of the series:

$$\begin{aligned} &\int_0^t F[\alpha(t-\tau_1),\tau_1]\dot{\alpha}(t-\tau_1)\,\mathrm{d}\tau_1 \\ &\simeq \int_0^{t_a} F[\alpha(t_a-\tau_1),\tau_1]\dot{\alpha}(t_a-\tau_1)\,\mathrm{d}\tau_1 \\ &\simeq -\dot{\alpha}(t_a)[I_1(0) + 0.5J_1(0)\alpha^2(t_a)] \end{aligned} \tag{13}$$

Comparing the last two lines of Eq. (13), the integral is dominated by the region near the lower limit ($\tau_1 = 0$). This is to be expected on physical grounds since small τ_1 corresponds to the recent past and the asymptotic nature of the series requires the motion to be slow in some sense. This suggests that in the integrand τ_1 can be neglected compared to t_a, i.e.,

$$\int_0^{t_a} F[\alpha(t_a-\tau_1),\tau_1]\dot{\alpha}(t_a-\tau_1)\,\mathrm{d}\tau_1 \simeq \dot{\alpha}(t_a)\int_0^{t_a} F[\alpha(t_a),\tau_1]\,\mathrm{d}\tau_1 \tag{14}$$

Finally, recalling that $\mathrm{I}_1(t_a)$ and $\mathrm{J}_1(t_a)$ are negligible and

$$F[\alpha(t_a),\tau_1] = F_0(0,\tau_1) + 0.5F_2(0,\tau_1)\alpha^2(t_a)$$

the last lines of Eqs. (13) and (14) are easily shown to be identical.

Thus, to first order, lift due to alpha dot is proportional to the area under the deficiency function *evaluated at the instantaneous angle of attack*, and thus

$$C_{L_{\dot{\alpha}}}[\alpha(t)] = -\int_0^{t_a} F[\alpha(t_a),\tau_1]\,\mathrm{d}\tau_1 = \mathrm{I}_1(0) + 0.5\mathrm{J}_1(0)\alpha^2(t_a)$$

In keeping with the spirit of the Taylor series interpretation of the stability derivative concept, the following definitions are perhaps preferable:

$$C_{L_{\dot{\alpha}}} = \mathrm{I}_1(0)$$

$$C_{L_{\dot{\alpha}\alpha\alpha}} = 0.5\frac{\partial^3 C_L}{\partial\dot{\alpha}\,\partial\alpha^2} = 0.5\mathrm{J}_1(0)$$

Of course, additional information is obtained by including higher-order terms in the series expansions. For example, the second-order terms reveal that alpha-double-dot effects are related to the second integral of the deficiency function again evaluated at $\alpha(t_a)$, as shown by Eqs. (15):

$$C_{L_{\ddot{\alpha}}} = \mathrm{I}_2(0) \qquad \text{and} \qquad C_{L_{\ddot{\alpha}\alpha\alpha}} = 0.5\mathrm{J}_2(0) \tag{15}$$

However, both an additional term and an unexpected identity appear:

$$2C_{L_{\ddot{\alpha}\alpha\alpha}} \equiv C_{L_{\alpha\dot{\alpha}\dot{\alpha}}} = 0.5\mathrm{J}_2(0)$$

Thus, dynamic stability derivatives and the dynamic terms of the asymptotic expansions, terms involving $\mathrm{I}_n(0)$ and $\mathrm{J}_n(0)$, are intimately related. When the asymptotic approximation to the superposition integral breaks down, the stability derivative concept does also. For harmonic motion and deficiency function forms given by Eqs. (5a) and (5b), this occurs at frequencies at or above the largest indicial function time constant as shown in the preceding section. When nonlinearities introduce harmonics, the harmonic frequency should be compared to the time constant for the corresponding nonlinear component of the deficiency function.

Relationship to Steady-State Oscillatory Data

Direct measurement of nonlinear indicial responses is difficult at best. Furthermore, suitable parameter identification techniques to extract them from experimental data have not been discussed in the literature. However, an assessment of the usefulness of approximating the indicial-response functional in terms of onset parameters [Eq. (4)] needs to be accomplished for more general cases than studied here. The limited objective of identifying significant onset parameters from steady-state oscillatory data is discussed in this section.

Consider again a harmonic oscillation about a constant mean angle of attack, given by Eq. (10a). Substituting Eqs. (10b) and (10c) into Eq. (9), the steady-state lift response has the form

$$C_L = C_{L_0} = \mathrm{G}_1\cos\Omega + \mathrm{G}_2\cos 2\Omega + \mathrm{G}_3\cos 3\Omega + \mathrm{H}_1\sin\Omega + \mathrm{H}_2\sin 2\Omega + \mathrm{H}_3\sin 3\Omega \tag{16}$$

The time-invariant response arises from nonzero values of α_0 and is simply the average of the maximum and minimum quasisteady lift values:

$$C_{L_0} = \alpha_0\{[(2\alpha_0^2 + 3A^2)/12]C_{L_{\alpha\alpha\alpha}} + C_{L_\alpha}(0,\infty)\}$$

From Eq. (9), it can be shown that all of the in-phase contributions, G_1 through G_3, are power series in k and contain a frequency independent quasisteady term. For example,

$$\mathrm{G}_1 = A\{C_{L_\alpha}(0,\infty) - \mathrm{I}_2k^2 + \mathrm{I}_4k^4 - \cdots\} + (B_1/2)\{C_{L_{\alpha\alpha\alpha}} - \mathrm{J}_2k^2 + \mathrm{J}_4k^4 - \cdots\} \tag{17a}$$

$$\mathrm{G}_2 = (B_2/4)\{C_{L_{\alpha\alpha\alpha}} - 4\mathrm{J}_2k^2 + 16\mathrm{J}_4k^4 - \cdots\} \tag{17b}$$

where I_n and J_n are evaluated at $\tau_1 = 0$.

The out-of-phase components are similar except that the leading terms are first order in frequency and, as expected, contain no quasisteady terms; e.g.,

$$\mathrm{H}_2 = -(B_2/2)k\{\mathrm{J}_1 - 4\mathrm{J}_3k^2 + \cdots\} \tag{17c}$$

Now, suppose that additional terms are needed in Eq. (4) to represent adequately the indicial response. For example, if onset rate is important and its effect varies linearly with angle of attack, terms proportional to alpha dot and the product of alpha and alpha dot should be considered. That is, two additional series are introduced in Eq. (9) involving the motion variables:

$$g_3(t) \equiv \dot{\alpha}^2 \qquad \text{and} \qquad g_4(t) \equiv \alpha\dot{\alpha}^2$$

From Eq. (10a), g_3 contributes a time-invariant term and a first harmonic. Similarly, g_4 contributes a constant term and components at the fundamental, first, and second harmonic frequencies. Thus, including these terms leaves the form of Eq. (16) undisturbed; however, its coefficients (G_1,H_1, etc.) then contain contributions from each of the g_i. As shown in Eqs. (17a–c), the effects of A and α_0 are separable from frequency effects; i.e., they may written in the form

$$\mathrm{G}_j = G_{0,j}c_{0,j} + G_{2,j}c_{2,j} + G_{3,j}c_{3,j} + G_{4,j}c_{4,j} \tag{18}$$

where

$$G_{i,j} = G_{i,j}(A,\alpha_0), \quad c_{i,j} = c_{i,j}(k); \qquad j = 1,2,3$$

and i corresponds to onset parameter g_i.

Individual effects due to each of the three nonlinear g_i on mean lift $G_{i,j}$ and $H_{i,j}$ are summarized in Table 1. Note that each onset parameter has a unique signature and a test matrix can be designed to accentuate the differences.

Table 1 Influence of onset parameters

Coefficient	g_2	g_3	g_4
C_{L_0}	0	A^2	$A^2\alpha_0$
G_1,H_1	$A^3 + 4A^2\alpha_0$	0	A^3
G_2,H_2	$A^2\alpha_0$	A^2	$A^2\alpha_0$
G_3,H_3	A^3	0	A^3

Given a candidate set of onset parameters, the identification problem consists of solving for the unknown $c_{i,j}$ on the right side of Eq. (18). Thus, a relatively simple harmonic analysis of data obtained by conventional techniques is sufficient to identify active onset parameters. Finally, note that this analysis requires no assumptions about the *form* of deficiency-function time history. It *does* assume, however, that no bifurcations occur within the test range and requires that the series in Eq. (9) converge.

Concluding Remarks

Earlier, the nonlinear indicial response was introduced by Tobak, Chapman, and Schiff as a means of modeling aerodynamic responses in nonlinear flight mechanics problems. They have also suggested possible simplifications based on relating indicial response to motion variables (and their derivatives) evaluated at step onset.

In this paper, the implications of using a Taylor series expansion (with respect to onset parameters) of the deficiency function are examined. A particularly simple expansion is shown to be applicable to thin two-dimensional airfoils with wake distortion nonlinearities. Furthermore, given the series approximation to the indicial response, the generalized superposition integral can be approximated by an asymptotic expansion, valid for sufficiently slow motions. For harmonic motion, the expansions diverge for frequencies greater than the slowest varying exponential involved in the corresponding deficiency function. The combined approximations, for the indicial response function and for the superposition integral, lead directly to relationships between conventional stability derivatives and indicial response characteristics. In the absence of bifurcation, these relationships can be used to identify dominant onset parameters from steady-state oscillatory force and moment data.

Further work is needed, especially concerning bifurcation of steady-state responses to step inputs due to their importance to aerodynamic hysteresis. The usefulness of representing the indicial response as a function of onset motion parameters depends on obtaining sufficient accuracy with a very limited number of terms. Assessments based on experimental data at realistic reduced frequencies should be made.

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References

[1]Tobak, M., and Chapman, G. T., "Nonlinear Problems in Flight Mechanics Involving Aerodynamic Bifurcations," *Unsteady Aerodynamics—Fundamentals and Applications to Aircraft Dynamics*, AGARD CP-386, Paper 25, May 1985.

[2]Tobak, M., and Schiff, L. B., "Aerodynamic Mathematical Modeling—Basic Concepts," AGARD Lecture Series 114, Paper 1, March 1981.

[3]Graham, G., and Yeow, K., "Two-Dimensional Post Stall Maneuver of a NACA 0015 Airfoil at High Pitching Rates," AIAA Paper 90-2810, Aug. 1990.

[4]Scott, M. T., and McCune, J. E., "Nonlinear Aerodynamics of Two-Dimensional Airfoils in Severe Maneuver," AIAA Paper 88-0129, Jan. 1988.

[5]Fung, Y. C., *An Introduction to the Theory of Aeroelasticity*, Dover, New York, 1969, p. 207.

[6]Etkin, B., *Dynamics of Atmospheric Flight*, Wiley, New York, 1972, pp. 165–168.

[7]Rugh, W. J., *Nonlinear System Theory*, Johns Hopkins Univ. Press, Baltimore, MD, 1981, pp. 53–73.

[8]Etkin, B., *Dynamics of Flight*, Wiley, New York, 1959, pp. 181, 182.

[9]Leishman, J. G., "Validation of Approximate Indicial Aerodynamic Functions for Two-Dimensional Subsonic Flow," *Journal of Aircraft*, Vol. 25, No. 10, 1988, pp. 914–922.

[10]Hanff, E. S., "Dynamic Nonlinear Airloads Representation and Measurement," *Unsteady Aerodynamics—Fundamentals and Applications to Aircraft Dynamics*, AGARD CP-386, Paper 27, May 1985.

[11]Tobak, M., "On the Minimization of Airplane Responses to Random Gusts," NACA TN-3290, Oct. 1957.

[12]Etkin, B., *Dynamics of Atmospheric Flight*, Wiley, New York, 1972, pp. 158, 159.